Section 3.3C: $\mathbb{R}$-semiconjugate maps and chaos.

The purpose of this segment is to investigate the occurrence of chaotic behavior for the semiconjugates of interval maps. This material is largely complementary to that in Segment B, since in most cases below, complicated behavior occurs without the existence of snap-back repellers, expanding fixed points, etc. In particular, we find that sensitivity to initial conditions is a property that extends from the real factor to the higher dimensional map (at least in the important case where the set D is compact). The same is not true about the existence of periodic points (Example 3.3.2) or snap-back repellers (Example 3.3.1), even when the factor map does possess such objects.

For simplicity, and to avoid certain types of undesirable possibilities, *in all the results of this section we assume that all semiconjugate links are bending everywhere* (see Definition 3.2.2).

Example 3.3.1. (The radial Logistic) Let us consider a special case of the radial map of Example 3.2.9 in dimension 2:

$$F(x,y) \doteq a(1-\alpha x-\beta y)[x,y], \quad a,\alpha,\beta>0$$

With $H(x,y) \doteq \alpha x+\beta y$, F is semiconjugate to $\phi(t)=at(1-t)$, which exhibits chaotic behavior in the interval [0,1] if the parameter a is sufficiently close to (and less than) 4. If

$$D \doteq \{(x,y) : \alpha x+\beta y \le 1\} \cap [0,\infty)^2$$

then $H(D)=[0,1]$. Since each fiber intersects a given ray in a single point, the domain of ϕ is in one to one correspondence with the points of each ray.

In fact, let R be the part of the ray through (x_0,y_0) that lies within the triangle D and let $t_0=y_0/x_0$ (assume that $x_0\neq 0$; the case $x_0=0$ is treated in a similar way). The restriction of F to R is topologically conjugate to ϕ; this is seen as follows: R is a line segment with one end point at the origin and the other at the point of intersection of the ray with the boundary line $\alpha x+\beta y=1$ of D. This point is (x',t_0x') where $x'=1/(\alpha+\beta t_0)$. Now, setting

$$s \doteq (\alpha+\beta t_0)x$$

it follows that as x ranges from 0 to x', the parameter s ranges from 0 to 1. The restriction of F to R is easily seen to be

$$F(x,t_0x)=as(1-s)[1,t_0]/(\alpha+\beta t_0)=as(1-s)c[x_0,y_0]$$

where $c \doteq 1/(\alpha x_0+\beta y_0)$. Now, note that the mapping $h(t) \doteq tc[x_0,y_0]$ is a homeomorphism of [0,1] onto R, and

$$h(\phi(s))=F(x,t_0x)=F\left(\frac{s}{\alpha+\beta t_0},\frac{t_0 s}{\alpha+\beta t_0}\right)=F(h(s))$$

i.e., the restriction of F to R is conjugate to ϕ on [0,1].

We may now conclude that the chaotic behavior of ϕ is duplicated by F on each ray through the origin; i.e., all the conclusions of Theorem 3.3.2 hold on each fixed ray. In particular, F has periodic points of all periods if $3.84 \leq a \leq 4$, although neither the origin, nor any of the (non-isolated) fixed points comprising the unstable invariant fiber H_p^{-1} where $p = 1 - 1/a$ can be snap-back repellers since the required derivative conditions do not hold. In fact, the fixed points in H_p^{-1} are not even expanding, since they are not isolated. More precisely, if $(\bar{x}, \bar{y}) \in H_p^{-1}$, i.e., $\alpha\bar{x} + \beta\bar{y} = p$, then with $a(1-p) = 1$ from ϕ,

$$DF(\bar{x}, \bar{y}) = \begin{bmatrix} 1 - a\alpha\bar{x} & -a\beta\bar{x} \\ -a\alpha\bar{y} & 1 - a\beta\bar{y} \end{bmatrix}$$

which has eigenvalues

$$\lambda_1 = 1, \quad \lambda_2 = 2 - a$$

i.e., the fixed points in H_p^{-1} are not expanding.

Theorem 3.3.3. *Let F be a (D, H, ϕ)-semiconjugate map, with $H \in C^1(D, \mathbb{R})$ and $D \subset \mathbb{R}^m$ a compact and convex set. If there are $p, q \in H(D)$ satisfying* (T3.3.1a) *for ϕ, then for each $x \in H_p^{-1}$ and $y \in H_q^{-1}$,*

$$\limsup_{n\to\infty} \|F^n(x) - F^n(y)\| > 0. \qquad \text{(T3.3.3a)}$$

In particular, if ϕ has a scrambled set S, then trajectories starting in H_S^{-1} cannot converge to periodic points of F.

Proof. By the MV Theorem 1.3(b), for all $u, v \in D$ and every $n \geq 1$,

$$|H(F^n(u)) - H(F^n(v))| \leq \|F^n(u) - F^n(v)\| \sup_{w\in L(u,v)} \|\nabla H(w)\|$$

where $L(u, v)$ is the line segment that joins $F^n(u)$ to $F^n(v)$. Since D is convex, $L(u, v) \subset D$, so that

$$\sup_{w\in L(u,v)} \|\nabla H(w)\| \leq \sigma \doteq \sup_{z\in D} \|\nabla H(z)\|.$$

Note that $0 < \sigma < \infty$. Now for each pair x, y as in the statement of the theorem, $H(x) = p$ and $H(y) = q$, so we obtain

$$\begin{aligned} \|F^n(x) - F^n(y)\| &\geq \frac{1}{\sigma} |H(F^n(x)) - H(F^n(y))| \\ &= \frac{1}{\sigma} |\phi^n(p) - \phi^n(q)|. \end{aligned}$$

The proof of (T3.3.3a) is complete upon taking limit supremum and using (T3.3.1a) for ϕ.

Next, suppose that ϕ has a scrambled set $S \subset H(D)$. If y is a periodic point of F, then there is a positive integer k such that

$$\phi^k(H(y)) = H(F^k(y)) = H(y)$$

Thus, $H(y)$ is a periodic point of ϕ, i.e., $H(y) \notin S$ and the preceding results apply.

Corollary 3.3.1. *Let F be a (D, H, ϕ)-semiconjugate map, with $H \in C^1(D, \mathbb{R})$ and $D \subset \mathbb{R}^m$ a compact and convex set. If ϕ has sensitive dependence on initial conditions, then so does F.*

Proof. Suppose that $x \in D$ and $\varepsilon > 0$, and let $t \doteq H(x) \in H(D)$. The set $H(B_\varepsilon(x) \cap D)$ is a nontrivial subinterval of $H(D)$ containing t. Due to the sensitivity of ϕ, there is $s \in H(B_\varepsilon(x) \cap D)$ such that

$$\mu \doteq \limsup_{n\to\infty} |\phi^n(t) - \phi^n(s)| > 0.$$

Now if we choose $y \in H_s^{-1} \cap B_\varepsilon(x)$, then Theorem 3.3.3 implies that

$$\limsup_{n\to\infty} \|F^n(x) - F^n(y)\| \geq \frac{\mu}{\sigma}$$

which proves that F has sensitive dependence on initial conditions.

Absent from Theorem 3.3.3 is the existence of periodic points that were so prominent in Theorems 3.3.1 and 3.3.2. Periodic points do not generally occur for maps in higher dimensions, as the next example illustrates using an $\mathbb{R}$-semiconjugate map.

Example 3.3.2. (The turning logistic) Let

$$D = \{(\rho, \theta) : \rho \in [0, 1], \theta \in \mathbb{R}\}$$

be the unit disk in $\mathbb{R}^2$ and define $F \in C(D)$ as

$$F(\rho, \theta) \doteq [a\rho(1-\rho), \theta + \alpha], \quad 3 < a < 4, 0 \leq \alpha < 2\pi.$$

Note that F is semiconjugate to the logistic map $f(\rho) \doteq a\rho(1-\rho)$ under the link map $H(\rho, \theta) \doteq \rho$. The addition of θ in effect bends each trajectory of f when $\alpha \neq 0$ so that the corresponding trajectory of F is essentially the time series of f that instead of oscillating in a horizontal strip, it now wraps around the center of the disk within an annulus

$$D_1 \doteq \{(\rho, \theta) : \rho \in [\mu, \gamma], \theta \in \mathbb{R}\}, \quad \gamma \doteq \frac{a}{4}, \mu \doteq a\gamma(1-\gamma),$$

Figure E3.3.2 shows a plot of the annulus D_1 in polar coordinates (the ring with thick borders), and the computer generated points of a single trajectory of F starting from a point inside D_1.