A problem on subsequences (posed by G. Ladas). Assume that $\{x_n\}_{n=0}^{\infty}$ is a sequence of real numbers satisfying the following conditions:

(i) There exists a subsequence $\{x_{i_n}\}_{n=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ which converges to zero,

(ii) For every subsequence $\{x_{k_n}\}_{n=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ which converges to zero, the subsequence of the preceding terms $\{x_{k_n-1}\}_{n=1}^{\infty}$ also converges to zero.

Question: Does the sequence $\{x_n\}_{n=0}^{\infty}$ converge to zero?

Answer: No, (i) and (ii) are not sufficient for the convergence of the entire sequence.

Counterexample. (H. Sedaghat) Define a sequence $\{x_n\}_{n=0}^{\infty}$ as follows:

$$x_n = \begin{cases} 1/(k-j+1) & \text{if } n = k^2 - j \text{ and } j = 0, 1, 2, \ldots, k \\ \text{arbitrary for all other values of } n & \end{cases}.$$

Various terms of $\{x_n\}_{n=0}^{\infty}$ are listed below:

$$\begin{aligned}
x_1 &= \frac{1}{2},\ x_0 = 1 \\
x_4 &= \frac{1}{3},\ x_3 = \frac{1}{2},\ x_2 = 1 \\
x_5 &= \text{arbitrary} \\
x_9 &= \frac{1}{4},\ x_8 = \frac{1}{3},\ x_7 = \frac{1}{2},\ x_6 = 1 \\
x_{11} &= \text{arbitrary},\ x_{10} = \text{arbitrary} \\
&\vdots \\
x_{k^2} &= \frac{1}{k+1},\ x_{k^2-1} = \frac{1}{k},\ \ldots\ x_{k^2-k} = 1 \\
x_{k^2+k-1} &= \text{arbitrary},\ x_{k^2+k-2} = \text{arbitrary}, \ldots,\ x_{k^2+1} = \text{arbitrary} \\
x_{(k+1)^2} &= \frac{1}{k+2},\ x_{(k+1)^2-1} = \frac{1}{k+1},\ \ldots\ x_{(k+1)^2-(k+1)} = 1 \\
&\vdots
\end{aligned}$$

The first few terms are

$$1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, x_5, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, x_{10}, x_{11}, \ldots$$

If the arbitrary terms $x_{k^2+1}, \ldots, x_{k^2+k-1}$ for each $k \geq 2$ are defined in such a way that no subsequence of theirs converges to zero then the sequence defined above satisfies (i) and (ii) but it may fail to converge to any real number.