The Higher Order Riccati Difference Equation

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1 Introduction

Let k be a positive integer and $a_0, a_1, \ldots, a_k$ be real numbers with $a_k \neq 0$. The rational difference equation

$$x_{n+1} = a_0 + \frac{a_1}{x_n} + \frac{a_2}{x_n x_{n-1}} + \cdots + \frac{a_k}{x_n \cdots x_{n-k+1}} \tag{1}$$

of order k can be transformed into a linear homogeneous difference equation

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + \cdots + a_k y_{n-k} \tag{2}$$

of order $k+1$ through a change of variables

$$x_n = \frac{y_n}{y_{n-1}}. \tag{3}$$

This relationship between equations (2) and (1) is similar to the relationship between the linear homogeneous differential equation of order two

$$u'' = b_1 u' + b_0 u$$

and the classical Riccati differential equation

$$v' = b_0 + b_1 v - v^2$$

which are linked via the change of variables $v = u'/u$; see, e.g., [4], [11]. For this reason we call Eq.(1) the *Riccati difference equation of order* k. Because the time index n is discrete, (1) is a discrete-time Riccati equation.

Equations (2) and (1) are related in a second, theoretically deeper fashion. The linear equation (2) is homogeneous of degree 1 over the group of nonzero real numbers under multiplication; see [7]. As such, it has a type-$(k, 1)$ order reduction via the inversion form symmetry characterized by the change of variables (3). The Riccati equation (1) is then just the semiconjugate factor of the linear equation (2). These assertions extend from $\mathbb{R}$ to arbitrary fields since both (2) and (1) can be defined over any nontrivial algebraic field $\mathcal{F}$. Certain special cases of Eq.(1) admit other types of form symmetries and can be reduced in order over $\mathcal{F}$ via semiconjugate factorization; see [9] for the basic theory. We discuss such a case below in which the analysis of a third order Riccati equation can reduced to the first order case.

2 Riccati, k = 2

The first order Riccati equation ($k = 1$) has been studied thoroughly (see [2], [3], [5], [6]) so we do not discuss that case here. In particular, Ladas and colleagues show in [2] and [6] how to transform the discrete Riccati equation of order one, defined initially as a linear fractional equation, into a linear difference equation of order two. The linear equation is then used to obtain detailed information about the solutions of the first order Riccati equation.

Using a similar approach in this section we obtain detailed information about the Riccati equation of order 2. In particular, we show that with non-negative coefficients almost all initial points in $\mathbb{R}^2$ generate solutions of the second order equation that converge to a positive fixed point, just like the first order case. However, unlike the first order case, there are also "rare" initial values generating periodic solutions of all possible periods as well as non-periodic, oscillatory solutions.

For easier reading, we use non-subscripted letters and write the second order Riccati difference equation as

$$x_{n+1} = a + \frac{b}{x_n} + \frac{c}{x_n x_{n-1}}. \tag{4}$$

In this section we show that even under non-negativity conditions on coefficients, i.e.,

$$a, b \geq 0,\ \ c > 0,\ \ x_0, x_{-1} \in (-\infty, \infty) \tag{5}$$

Eq.(4) exhibits all behavior types that are possible for the first order case ($c = 0$) for all real values of a, b. The material in this section is taken largely from [1].

The linear homogeneous equation of order 3 that is associated with

$$y_{n+1} = ay_n + by_{n-1} + cy_{n-2}. \tag{6}$$

If we define the initial values for (6) as

$$y_0 = x_0 y_{-1},\ \ y_{-1} = x_{-1} y_{-2} \text{ and set} \tag{7}$$
$$y_{-2} = 1 \text{ (or any fixed nonzero real number)}$$

then we obtain a one to one correspondence between the solutions of (4) and those solutions of (6) that do not contain zero; i.e., each solution of (4) uniquely defines a solution of (6) that does not pass through the origin and vice versa. If $\{y_n\}$ is a solution of (6) with $y_k = 0$ for some least k then $x_{k+1} = y_{k+1}/y_k$ is not defined. Under conditions (7) the correspondence between solutions of (6) that pass through the origin and those of (4) that become undefined is also one to one.

2.1 The singularity set

An initial point (x_{-1}, x_0) that generates a solution $\{y_n\}$ of (6) with $y_k = 0$ for some least k leads to an undefined value $x_{k+1} = y_{k+1}/y_k$. We call the set S of all such initial points the *singularity set* of Eq.(4). The singularity set is also called the "forbidden set" in the literature ([2], [6]). In order to determine the global behavior of all solutions of (4) under conditions (5) it is necessary to determine S.

We use the solutions of the linear equation (6) to determine S. The characteristic polynomial of (6) is

$$P(\lambda) = \lambda^3 - a\lambda^2 - b\lambda - c = 0. \tag{8}$$

Note that the real solutions of (8) also give the fixed points of Eq.(4). The cubic polynomial P has at least one real root. The next two results give more precise information about the roots of P.

Lemma 1 *Assume that conditions (5) hold. Then the polynomial P has precisely one positive real root ρ that satisfies*

$$\rho \geq \max\left\{\sqrt[3]{c}, \frac{a + \sqrt{a^2 + 4b}}{2}\right\} \tag{9}$$

with equality holding if and only if $a = b = 0$.

Proof. By the Descartes rule of signs P has only one positive root ρ under conditions (5). Further

$$P(\lambda) = \lambda(\lambda^2 - a\lambda - b) - c$$

and the roots of $\lambda^2 - a\lambda - b$ are $(a \pm \sqrt{a^2 + 4b})/2$. If λ_0 is the non-negative root then since $P(\lambda_0) = -c < 0$ it follows that $\rho > \lambda_0$. Next, from $P(\rho) = 0$ we obtain

$$\rho^2 - a\rho - b = \frac{c}{\rho} \tag{10}$$

which implies $c/\rho \leq \rho^2$, i.e., $\rho \geq \sqrt[3]{c}$. Finally, if equality holds in (9) then $\rho = \sqrt[3]{c} \neq \lambda_0$ since $P(\lambda_0) = -c \neq 0$. But then $P(\sqrt[3]{c}) = 0$ implies that $a\sqrt[3]{c} + b = 0$ which implies $a = b = 0$ because $a, b \geq 0$. Conversely if $a = b = 0$ then $\lambda^3 - c = 0$ so $\rho = \sqrt[3]{c}$ and equality holds in (9). ■

It is possible to find a formula for ρ in terms of radicals (see [10]) but we omit it since that information is not particularly useful here.

Lemma 2 *Assume that conditions (5) hold and let ρ be the positive root of (8).*

(a) Eq.(8) has two other roots that can be calculated in terms of ρ as

$$r^{\pm} = -\frac{\rho - a}{2} \pm \sqrt{\left(\frac{\rho + a}{2}\right)^2 - \rho^2 + b}. \tag{11}$$

(b) If $(\rho + a)^2 \geq 4(\rho^2 - b)$ then the real roots $r^{\pm}$ are negative and

$$-\rho < r^- \leq -\frac{\rho - a}{2} \leq r^+ < 0.$$

(c) If $(\rho + a)^2 < 4(\rho^2 - b)$ (e.g., if $b = 0$) then the complex roots $r^{\pm}$ satisfy

$$\left|r^{\pm}\right| = \sqrt{\rho^2 - a\rho - b} = \sqrt{\frac{c}{\rho}}. \tag{12}$$

Proof. (a) Dividing $P(\lambda)$ by $\lambda - \rho$ gives the quadratic polynomial $Q(\lambda) = \lambda^2 + (\rho - a)\lambda + \rho^2 - a\rho - b$. The two roots $r^{\pm}$ of Q are given by (11).

(b) In this case,

$$r^- > -\rho \quad \text{iff} \quad \rho - \frac{\rho - a}{2} > \sqrt{\left(\frac{\rho + a}{2}\right)^2 - \rho^2 + b} \quad \text{iff} \quad \rho^2 > b.$$

The last inequality is true by (10) in the proof of Lemma 1. Similarly,

$$r^+ < 0 \quad \text{iff} \quad \sqrt{\left(\frac{\rho + a}{2}\right)^2 - \rho^2 + b} < \frac{\rho - a}{2} \quad \text{iff} \quad \rho^2 - a\rho - b > 0.$$

The last inequality is true again by (10).

(c) In this case the moduli of $r^{\pm}$ are easily found to be given by (12). If $b = 0$ then since by (9) $\rho > a$ it follows that $(\rho + a)^2 < (2\rho)^2 = 4\rho^2$ and roots $r^{\pm}$ are complex. ■

Based on Lemma 2 the next result summarizes the standard facts about the solutions of the linear equation (6). Of particular interest to us is the fact that the coefficients of solutions all have the same general formula.

Lemma 3 *Suppose that conditions (5) hold.*

(a) If $(\rho + a)^2 > 4(\rho^2 - b)$ *then for all* $n \geq 0$

$$y_n = C_1\rho^n + C_2(r^+)^n + C_3(r^-)^n$$

where the coefficients $C_j,\ j = 1, 2, 3$ *are given by*

$$C_j(x_0, x_{-1}) = \alpha_{1j}x_0x_{-1} + \alpha_{2j}x_{-1} + \alpha_{3j} \tag{13}$$

for suitable constants $\alpha_{ij},\ i, j = 1, 2, 3$ *that do not depend on the initial values.*

(b) If $(\rho + a)^2 = 4(\rho^2 - b)$ *then for all* $n \geq 0$

$$y_n = C_1\rho^n + (C_2 + C_3n)r^n \quad \text{where } r = r^+ = r^- = -\frac{\rho - a}{2}$$

where the coefficients C_j *are given by (13) with constants* $\alpha_{ij},\ i, j = 1, 2, 3$ *appropriate to this case.*

(c) If $(\rho + a)^2 < 4(\rho^2 - b)$ *then for all* $n \geq 0$

$$y_n = C_1\rho^n + (\rho^2 - a\rho - b)^{n/2}(C_2 \cos n\theta + C_3 \sin n\theta)$$

where $\theta \in (\pi/2, \pi)$ *is a constant and the coefficients* C_j *are given by (13) with constants* α_{ij}*,* $i, j = 1, 2, 3$ *appropriate to this case.*

Proof. The solutions $\{y_n\}$ in each case are obtained routinely from the basic linear theory so we only explain about (13) and the range of θ in (c).

(a) The coefficients C_j satisfy the system

$$C_1 + C_2 + C_3 = x_0x_{-1},\ C_1/\rho + C_2/(r^+) + C_3/(r^-) = x_{-1},$$
$$\text{and} \quad C_1/\rho^2 + C_2/(r^+)^2 + C_3/(r^-)^2 = 1.$$

This system which is linear in the C_j can be easily solved to obtain

$$C_1 = \frac{\rho^2[x_0x_{-1} - (r^+ + r^-)x_{-1} + r^+r^-]}{(\rho - r^+)(\rho - r^-)} \tag{14}$$

from which we can read off the values of the constants α_{1j}. Further,

$$C_2 = \frac{-(r^+)^2\left[x_0x_{-1} - (r^+ + r^-)x_{-1} + r^+r^-\right]}{(\rho - r^+)(r^+ - r^-)}$$

gives the constants α_{2j} and

$$
\begin{aligned}
C_3 &= x_0 x_{-1} - C_1 - C_2 \\
&= (1 - \alpha_{11} - \alpha_{12}) x_0 x_{-1} - (\alpha_{21} + \alpha_{22}) x_{-1} - (\alpha_{31} + \alpha_{32})
\end{aligned}
$$

which gives α_{3j}.

(b) In this case the coefficients C_j satisfy

$$
\begin{gathered}
C_1 + C_2 = x_0 x_{-1},\ C_1/\rho - 2(C_2 - C_3)/(\rho - a) = x_{-1}, \\
\text{and} \quad C_1/\rho^2 + 4(C_2 - 2C_3)/(\rho - a)^2 = 1.
\end{gathered}
$$

From these we obtain

$$
C_1 = \frac{4\rho^2(\rho - a)}{(3\rho - a)^2} x_0 x_{-1} + \frac{4\rho^2}{(\rho - a)(3\rho - a)} x_{-1} + \frac{\rho^2(\rho - a)^2}{(3\rho - a)^2} \tag{15}
$$

from which we can read off the values of the constants α_{1j}. Further,

$$
C_2 = x_0 x_{-1} - C_1 = (1 - \alpha_{11}) x_0 x_{-1} - \alpha_{21} x_{-1} - \alpha_{31}
$$

gives the constants α_{2j} for this case and

$$
C_3 = \frac{\rho - a}{2} x_{-1} - \frac{\rho - a}{2\rho} C_1 + C_2
$$

from which α_{3j} can be calculated.

(c) In this case the coefficients C_j satisfy

$$
\begin{gathered}
C_1 + C_2 = x_0 x_{-1},\ C_1/\rho + (C_2 \cos\theta - C_3 \sin\theta)/\sqrt{\rho^2 - a\rho - b} = x_{-1}, \\
\text{and} \quad C_1/\rho^2 + (C_2 \cos 2\theta - C_3 \sin 2\theta)/(\rho^2 - a\rho - b) = 1
\end{gathered} \tag{16}
$$

where θ is defined by the equalities

$$
\cos\theta = -\sqrt{\frac{\rho}{c}} \frac{\rho - a}{2}, \quad \sin\theta = \sqrt{\frac{\rho}{c}} \sqrt{\rho^2 - b - \left(\frac{\rho + a}{2}\right)^2} \tag{17}
$$

which also show that $\theta \in (\pi/2, \pi)$. From (16) we obtain using $\rho^2 - a\rho - b = c/\rho$,

$$
C_1 = \frac{\rho^2 c}{\rho^3 + c - 2\sqrt{\rho^3 c} \cos\theta} \left[\frac{\rho}{c} x_0 x_{-1} - 2\sqrt{\frac{\rho}{c}} (\cos\theta) x_{-1} + 1 \right] \tag{18}
$$

from which we can read off the values of the constants α_{1j}. Further,

$$C_2 = x_0 x_{-1} - C_1$$
$$C_3 = \frac{c \sin 2\theta}{\rho^3} C_1 + \frac{\cos 2\theta}{\sin 2\theta} C_2 - \frac{c}{\rho} \sin 2\theta$$

from which α_{ij}, $i = 2, 3$ can be calculated. ■

Since each of the coefficients C_j depends on the two intial values, each solution of the linear equation (6) is a function $y_n(u, v)$ of two variables, all other parameters being fixed. Thus the singularity set S of Eq.(4) can be written as

$$S = \bigcup_{n=-1}^{\infty} \{(u, v) : y_n(u, v) = 0\}.$$

Note that $S \subset \mathbb{R}^2 \backslash (0, \infty)^2$ because under conditions (5) each solution $\{x_n\}$ of (4) with $(x_0, x_{-1}) \in (0, \infty)^2$ satisfies $x_n > 0$ for all $n \geq -1$ and thus there are no undefined values. Now the next result is an immediate consequence of Lemma 3.

Theorem 4 *Suppose that conditions (5) hold. Then the singularity set of Eq.(4) is the following sequence of hyperbolas*

$$S = \bigcup_{n=-1}^{\infty} \{(u, v) : \beta_{1n} uv + \beta_{2n} v + \beta_{3n} = 0\} \subset \mathbb{R}^2 \backslash (0, \infty)^2$$

where the sequences β_{in} are defined as follows:

(a) If $(\rho + a)^2 > 4(\rho^2 - b)$ then

$$\beta_{in} = \alpha_{i1} + \alpha_{i2} (r^+/\rho)^n + \alpha_{i3} (r^-/\rho)^n$$

where α_{ij} are the constants in Lemma 3(a).

(b) If $(\rho + a)^2 = 4(\rho^2 - b)$ then

$$\beta_{in} = \alpha_{i1} + (-1/2)^n (1 - a/\rho)^n (\alpha_{i2} + \alpha_{i3} n)$$

where α_{ij} are the constants in Lemma 3(b).

(c) If $(\rho + a)^2 < 4(\rho^2 - b)$ then

$$\beta_{in} = \alpha_{i1} + \left(c/\rho^3\right)^{n/2} (\alpha_{i2} \cos n\theta + \alpha_{i3} \sin n\theta)$$

where α_{ij} are the constants in Lemma 3(c).

2.2 Global asymptotic stability

In this section we use the preceding results to show that under conditions (5) almost all solutions of Eq.(4) converge to the positive fixed point ρ if at least one of the parameters a or b is positive.

Lemma 5 *Under conditions (5) ρ is the unique positive fixed point of (4) and if $a+b>0$ then ρ is locally asymptotically stable.*

Proof. Define

$$f(u,v)=a+\frac{b}{u}+\frac{c}{uv}.$$

Since the fixed points of (4) correspond to the roots of the polynomial P in (8), the uniqueness of ρ follows from Lemma 1. Next, the characteristic equation of the linearization of (4) at the fixed point (ρ,ρ) is

$$\lambda^2-f_u(\rho,\rho)\lambda-f_v(\rho,\rho)=0 \tag{19}$$

where

$$f_u=\frac{-1}{u^2}\left(b+\frac{c}{v}\right),\quad f_v=\frac{-c}{uv^2}.$$

These and the fact that $b\rho+c=\rho^3-a\rho^2$ determine Eq.(19) as

$$\lambda^2+\frac{\rho-a}{\rho}\lambda+\frac{c}{\rho^3}=0.$$

The zeros of this quadratic are

$$\lambda^{\pm}=\frac{\rho-a}{2\rho}\left[-1\pm\sqrt{1-\frac{4c}{\rho\left(\rho-a\right)^2}}\right].$$

If

$$\rho\left(\rho-a\right)^2\geq 4c \tag{20}$$

then the numbers $\lambda^{\pm}$ are real and $\lambda^-\leq\lambda^+<0$. Further, a little algebra shows that $\lambda^->-1$ if and only if

$$\sqrt{1-\frac{4c}{\rho\left(\rho-a\right)^2}}<\frac{\rho+a}{\rho-a}$$

which is obviously true since the left side is less than 1 and the right side greater than 1. Thus if (20) holds then ρ is a stable node for (4). Next suppose that (20) is false. Then $\lambda^{\pm}$

are complex with $|\lambda^{\pm}| = \sqrt{c/\rho^3} < 1$ where the inequality holds by Lemma 1 when $a+b>0$. Thus if (20) is false then ρ is a stable focus for (4). These cases exhaust all possibilities so ρ is locally asymptotically stable. ■

In considering the global behavior of solutions of Eq.(4) the following set must be considered:

$$M = S \cup \{(u,v) : C_1(u,v) = 0\} = S \cup \{(u,v) : \alpha_{11}uv + \alpha_{21}v + \alpha_{31} = 0\} \tag{21}$$

where F is the forbidden set of (4) as determined in Theorem 4 and α_{i1} are the constants defined in Lemma 3.

Theorem 6 *Assume that conditions (5) hold with $a+b>0$. Then the positive fixed point ρ is globally asymptotically stable relative to $\mathbb{R}^2\backslash M$ where the set $M \subset \mathbb{R}^2\backslash(0,\infty)^2$ defined by (21) has Lebesgue measure zero.*

Proof. By Lemma 5 ρ is stable so it only remains to prove global attractivity. If $\{x_n\}$ is a solution of Eq.(4) then we claim that $\lim_{n\to\infty} x_n = \rho$ if $(x_0, x_{-1}) \notin M$.

First, consider the case where $r^{\pm}$ are real and distinct. In this case, Lemma 3 implies that

$$x_n = \frac{y_n}{y_{n-1}} = \frac{C_1\rho^n + C_2(r^+)^n + C_3(r^-)^n}{C_1\rho^{n-1} + C_2(r^+)^{n-1} + C_3(r^-)^{n-1}}. \tag{22}$$

Since $(x_0, x_{-1}) \notin M$ we have $C_1 = C_1(x_0, x_{-1}) \neq 0$. Now dividing by $C_1\rho^{n-1}$ yields

$$x_n = \frac{\rho + (C_2\rho/C_1)(r^+/\rho)^n + (C_3\rho/C_1)(r^-/\rho)^n}{1 + (C_2/C_1)(r^+/\rho)^{n-1} + (C_3/C_1)(r^-/\rho)^{n-1}}$$

which implies, by Lemma 2(b), that $\lim_{n\to\infty} x_n = \rho$. Next, in the case of equal real roots a similar calculation gives

$$x_n = \frac{\rho + r(C_2/C_1 + C_3 n/C_1)(r/\rho)^{n-1}}{1 + [C_2/C_1 + C_3(n-1)/C_1](r/\rho)^{n-1}}$$

Since by Lemma 2(b) $|r/\rho| < 1$ it follows that $\lim_{n\to\infty} x_n = \rho$. Next, in the case of complex roots

$$x_n = \frac{\rho + \sqrt{\rho^2 - a\rho - b}(1 - a/\rho - b/\rho^2)^{(n-1)/2}(C_2\cos n\theta + C_3\sin n\theta)/C_1}{1 + (1 - a/\rho - b/\rho^2)^{(n-1)/2}[C_2\cos(n-1)\theta + C_3\sin(n-1)\theta]/C_1}$$

so clearly $\lim_{n\to\infty} x_n = \rho$.

Finally, since M is a countable collection of hyperbolas it has Lebesgue measure zero in $\mathbb{R}^2$. To establish that $M \subset \mathbb{R}^2 \backslash (0,\infty)^2$ it remains to show that the set

$$\{(u,v) : C_1(u,v) = 0\} = \{(u,v) : \alpha_{11}uv + \alpha_{21}v + \alpha_{31} = 0\} \tag{23}$$

does not intersect the positive quadrant $(0,\infty)^2$. From expressions (14), (15) and (18) above we see that $\alpha_{i1} > 0$ for $i = 1,2,3$ in each of the three possible cases. Thus the set (23) cannot contain points (u,v) with $u, v > 0$. ■

In the boundary case $a = b = 0$ in (5) Theorem 6 is false; as the next proposition shows the solutions of (4) exhibit a completely different behavior in this case.

Proposition 7 *If neither of the initial values x_0, x_{-1} is zero then the corresponding solution of*

$$x_{n+1} = \frac{c}{x_n x_{n-1}}, \quad c \neq 0 \tag{24}$$

is given as

$$\left\{ x_{-1}, x_0, \frac{c}{x_0 x_{-1}}, x_{-1}, x_0, \frac{c}{x_0 x_{-1}}, \ldots \right\}.$$

In particular, every non-constant solution of (24) with $(x_0, x_{-1}) \neq (\sqrt[3]{c}, \sqrt[3]{c})$ has period 3.

The next result applies Theorem 6 to an equation that is similar to (4).

Corollary 8 *Assume that conditions (5) hold for the following equation*

$$z_{n+1} = \frac{1}{a + bz_n + cz_n z_{n-1}}. \tag{25}$$

If $a > 0$ and $z_0, z_{-1} \geq 0$ or if $a + b > 0$ and $z_0 > 0$, $z_{-1} \geq 0$ then $\lim_{n\to\infty} z_n = 1/\rho$ where ρ is defined in Lemma 1.

Proof. Since the change of variables $x_n = 1/z_n$ transforms (25) into (4), if $z_0, z_{-1} > 0$ then Theorem 6 implies that $\lim_{n\to\infty} z_n = \lim_{n\to\infty}(1/x_n) = 1/\rho$. If $a > 0$ and either $z_0 = 0$ or $z_{-1} = 0$ then from (25) we find that $z_1, z_2 > 0$ so again Theorem 6 applies. The last case is argued similarly. ■

2.3 Exceptional solutions: Oscillatory and convergent

The proof of Theorem 6 contains information about solutions that do not converge to ρ. These are exceptional solutions of (4) since they can only originate in the set $M \backslash S$ So the fact that they do occur in the second order case with positive parameters is an indication of the greater complexity of the higher order Riccati equation.

Theorem 9 *Assume that conditions (5) hold with $a+b>0$.*

(a) The hyperbola $H=\{(u,v): uv-(r^+ + r^-)v + r^+r^- = 0\}$ is an invariant subset of M.

(b) If $(\rho+a)^2 \geq 4(\rho^2-b)$ and $(x_0,x_{-1}) \in H\backslash\{(r^+,r^+)\}$ then $\lim_{n\to\infty} x_n = r^-$.

(c) Let $(\rho+a)^2 < 4(\rho^2-b)$. If $\theta=\pi q/p$ satisfies (17) for positive integers p,q that are relatively prime then for each $(x_0,x_{-1}) \in H$ the corresponding solution $\{x_n\}$ of (4) has period p. If θ is an irrational multiple of π then the corresponding solution of (4) is oscillatory but not periodic and the orbit $\{(x_n,x_{n-1})\}$ is dense in H.

Proof. (a) Notice from (14), (15) and (18) that the expression for C_1 is real even if $r^\pm$ are complex and that $C_1(x_0,x_{-1})=0$ if and only if

$$x_0x_{-1}-(r^+ + r^-)x_{-1} + r^+r^- = 0. \tag{26}$$

Indeed, from (18) we obtain $C_1=0$ if and only if

$$x_0x_{-1}-2\sqrt{\frac{c}{\rho}}(\cos\theta)x_{-1}+\frac{c}{\rho}=0$$

which is identical to (26) if $r^\pm$ are complex. Thus $C_1=0$ in all cases if it is shown that $C_1(x_{n+1},x_n)=0$ for all $n\geq 0$ whenever x_0 and x_{-1} satisfy (26).

Now

$$\begin{aligned} C_1(x_1,x_0) &= x_1x_0-(r^+ + r^-)x_0 + r^+r^- \\ &= x_0\left(a+\frac{b}{x_0}+\frac{c}{x_0x_{-1}}\right)-(r^+ + r^-)x_0 + r^+r^- \\ &= ax_0+b+\frac{c}{x_{-1}}+(\rho-a)x_0+\rho^2-a\rho-b \\ &= \frac{c}{x_{-1}}+\rho x_0+\rho^2-a\rho. \end{aligned} \tag{27}$$

If (26) holds then

$$x_0=-(\rho-a)-\frac{\rho^2-a\rho-b}{x_{-1}}=-(\rho-a)-\frac{c}{\rho x_{-1}}$$

which if inserted into (27) yields $C_1(x_1, x_0) = 0$. The proof of (a) can now be easily completed by induction.

(b) In this case the roots $r^\pm$ are real. First suppose that $(\rho + a)^2 > 4(\rho^2 - b)$. If $(x_0, x_{-1}) \in H$ then $C_1 = 0$ in (22) and thus

$$x_n = \frac{y_n}{y_{n-1}} = \frac{C_2(r^+)^n + C_3(r^-)^n}{C_2(r^+)^{n-1} + C_3(r^-)^{n-1}}.$$

If $C_3 = 0$ then $x_n = r^+$ for all n which can occur only if $x_1 = x_0 = r^+$. If $C_3 \neq 0$ then dividing by $C_3(r^-)^{n-1}$ and taking the limit gives

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{(C_2/C_3)(r^+/r^-)^n \bar{r} + r^-}{(C_2/C_3)(r^+/r^-)^{n-1} + 1} = r^-.$$

The argument for the case $(\rho + a)^2 = 4(\rho^2 - b)$ is similar but using Lemma 3(b); we omit the straightforward details.

(c) In this case the roots $r^\pm$ are complex and if $C_1 = 0$ then from Lemma 3 we obtain

$$\begin{aligned} x_n &= \frac{(\rho^2 - a\rho - b)(C_2 \cos n\theta + C_3 \sin n\theta)}{C_2 \cos(n-1)\theta + C_3 \sin(n-1)\theta} \\ &= \frac{c}{\rho}\cos\theta + \frac{c}{\rho}\sin\theta \frac{C_3\cos(n-1)\theta - C_2 \sin(n-1)\theta}{C_2 \cos(n-1)\theta + C_3 \sin(n-1)\theta}. \end{aligned}$$

Define $\cos\phi = C_2/\sqrt{C_2^2 + C_3^2}$ and $\sin\phi = C_3/\sqrt{C_2^2 + C_3^2}$. Then

$$\begin{aligned} x_n &= \frac{c}{\rho}\cos\theta + \frac{c}{\rho}\sin\theta \frac{\sin\phi\cos(n-1)\theta - \cos\phi\sin(n-1)\theta}{\cos\phi\cos(n-1)\theta + \sin\phi\sin(n-1)\theta} \\ &= \frac{c}{\rho}\cos\theta - \frac{c}{\rho}\sin\theta \frac{\sin[(n-1)\theta - \phi]}{\cos[(n-1)\theta - \phi]} \\ &= \frac{c}{\rho}\cos\theta - \frac{c}{\rho}\sin\theta \tan[(n-1)\theta - \phi]. \end{aligned} \tag{28}$$

Now if $\theta = \pi q/p$ is a rational multiple of π then it follows from (28) that x_n is periodic (with period p if q/p is in reduced form). If θ is not a rational multiple of π then the angles $(n-1)\theta - \phi$ form a dense subset of the circle as $n \to \infty$. Given that $\tan x$ is a homeomorphism from $(-\pi/2, \pi/2)$ to $\mathbb{R}$ we conclude from (28) that the sequence $\{x_n\}$ is dense in $\mathbb{R}$. Therefore, the orbit $\{(x_n, x_{n-1})\}$ is dense in H. ■

2.4 Negative coefficients

In the preceding sections we examined the global behavior of Eq.(4) subject to conditions (5). The natural question to ponder is what kinds of behaviors are possible for the solutions of (4) if conditions (5) do not hold. Numerical simulations suggest that oscillatory solutions (periodic or not) occur *non-exceptionally* so a greater variety of behaviors are observable.

We leave further exploration of Eq.(4) with $a, b, c \in \mathbb{R}$, $c \neq 0$ to future studies but note that for certain negative parameter values the cases discussed in the preceding sections are revisited in disguise as the next corollary shows.

Corollary 10 *Assume that $a, c < 0 \leq b$ in Eq.(4). If $x_0, x_{-1} \notin -M$ where M is the set defined in (21) then* $\lim_{n\to\infty} x_n = -\rho$.

Proof. Since setting $w_n = -x_n$ gives

$$w_{n+1} = -x_{n+1} = -a - \frac{b}{x_n} - \frac{c}{x_n x_{n-1}} = -a + \frac{b}{w_n} - \frac{c}{w_n w_{n-1}}$$

it follows that w_n satisfies (4) subject to conditions (5). The proof is completed by applying Theorem 6. ■

3 Riccati, k = 3

The general Riccati difference equation of order 3 can be stated as

$$x_{n+1} = a + \frac{b}{x_n} + \frac{c}{x_n x_{n-1}} + \frac{d}{x_n x_{n-1} x_{n-2}}. \tag{29}$$

If $d = 0$ then the second order case is obtained. An obvious approach to analyzing Eq.(29) when the coefficients a, b, c, d are real numbers is by studying the solutions of the fourth order analog of (6), i.e., the linear homogeneous equation that is associated with (29). This potentially rewarding study is left to the interested reader.

In this section we dicuss Eq.(29) in a general context so as to highlight some other features of higher order Riccati difference equations. Let F be a nontrivial field, i.e., with two or more elements. For notational simplicity, we denote the two operations of addition and multiplication in F by the usual ones for real numbers. Familiar examples of fields other than $\mathbb{R}$ include the complex numbers $\mathbb{C}$ and the finite fields $\mathbb{Z}_p$, i.e., the field of intergers modulo the prime number p. Let us assume that the coefficients and initial values of (29) satisfy

$$a, b, c \in F, \quad d, x_0, x_{-1}, x_{-2} \in F \backslash \{0\}. \tag{30}$$

The following general result is established by straightforward calculation. It may be compared with Proposition 7.

Proposition 11 *If the field F contains at least four distinct elements then every solution of Eq.(29) subject to conditions (30) has period 4 given by*

$$\left\{x_{-2},x_{-1},x_0,\frac{d}{x_0x_{-1}x_{-2}},x_{-2},x_{-1},x_0,\frac{d}{x_0x_{-1}x_{-2}},\ldots\right\}.$$

The period 4 is not necessarily prime. For instance, let $F=\mathbb{R}$. If $d>0$ and $x_0=x_{-1}=x_{-2}=\sqrt[4]{d}$ then $x_n=\sqrt[4]{d}$ for all $n\geq 1$. If $x_{-2}=x_0=\alpha$ and $x_{-1}=\sqrt{d}/\alpha$ then we have a solution of period 2:

$$\left\{\alpha,\frac{\sqrt{d}}{\alpha},\alpha,\frac{\sqrt{d}}{\alpha},\ldots\right\}.$$

The next result exploits the semiconjugate property of a special case of (29). It effectively reduces the order to one and offers a quick way of seeing that solutions of period 2 may occur non-exceptionally for Eq.(29); thus, there is a fundamental qualitative difference between equations (4) and (29) even over the field $\mathbb{R}$.

Proposition 12 *Let F be a nontrivial field. Then each solution $\{x_n\}$ of*

$$x_{n+1}=\frac{b}{x_n}+\frac{d}{x_nx_{n-1}x_{n-2}},\quad b\in F,\ d,x_0,x_{-1},x_{-2}\in F\backslash\{0\} \tag{31}$$

satisfies the following pair of lower order equations on F

$$t_{n+1}=b+\frac{d}{t_{n-1}},\quad t_0=x_0x_{-1},\ t_{-1}=x_{-1}x_{-2}, \tag{32}$$

$$x_{n+1}=\frac{t_{n+1}}{x_n}. \tag{33}$$

The proof of Proposition 12 easily follows from the general results in [8]. The second order equation (32) is the semiconjugate factor of (31) that is obtained via the identity form symmetry characterized by the change of variables $t_n=x_nx_{n-1}$. Eq.(33) is the cofactor equation of (31) which gives the solution $\{x_n\}$ through a first order nonautonomous equation that uses a related solution $\{t_n\}$ of (32).

Eq.(32) is a special type of second order equation because the odd and even terms of the solution $\{t_n\}$ satisfy the first order Riccati difference equation as follows:

$$t_{2n+2}=b+\frac{d}{t_{2n}},\quad t_{2n+1}=b+\frac{d}{t_{2n-1}}.$$

If $F = \mathbb{R}$ and $b, d > 0$ then by well-known results (see [2] and [6]) $t_{2n}, t_{2n-1} \to \alpha$ where α is the positive root of $t^2 - bt - d$. Hence $t_n \to \alpha$ and this results in the corresponding solution of (33) and thus also of (31) to converge to a solution of period 2. For $F = \mathbb{R}$ these assertions can also be proved using the fourth order linear homogeneous equation

$$y_{n+1} = by_{n-1} + dy_{n-3}$$

which has two complex and two real eigenvalues for $b, d > 0$. However, the sitatution can be very different in other fields as the next example shows.

Example 13 *Let $F = \mathbb{Z}_3$ and consider the first order Riccati difference equation*

$$r_{n+1} = b + \frac{d}{r_n}, \quad b \in \{0, 1, 2\}, \; d, x_0 \in \{1, 2\}. \tag{34}$$

Straightforward calculation shows that Eq.(34) can only have constant solutions if $b = 0$ and $d = 1$ ($r_n = 1$ or $r_n = 2$ for all n) or if $b = 1$ and $d = 2$ ($r_n = 2$ for all n). If $b = 0$ and $d = 2$ then every solution in $\mathbb{Z}_3$ has period 2 with range $\{1, 2\}$. There are no solutions of (34) for other values of b, d unlike the situation for $\mathbb{R}$.

4 Conlcuding remarks

It is also of interest to consider the solutions of the second order Riccati difference equation over an arbitrary field $\mathcal{F}$. What types of behavior are possible if $\mathcal{F} = \mathbb{C}$ the field of complex numbers? On the other hand, what if $\mathcal{F} = \mathbb{Z}_p$, the finite field of integers modulo the prime integer p?

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